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Dupin's Cyclide as a Self-dual Surface.

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Introduction.

A correlation between the points and the planes of space is determined when it is assumed that by it five arbitrary points are sent into five arbitrary planes. If this correlation sends the points of a given surface into the planes of the same surface, the latter will be said to be self-dual. Its order and class are then the same, and the singularities occurring in the surface considered as an envelope of planes are the dual of those found in the surface taken as a locus of points. The determination of a definite correlation which will render a particular surface self-dual is the chief object of this paper.

Dupin's Cyclide, the surface here considered, may be defined as a surface of the fourth order, with four double

points, and a double conic, which will be taken in the plane at infinity. (Salmon, Geometry of Three Dimensions, 4th Edition §§ 560, 567). To determine its equation we take a tetrahedron of reference with vertices at the four double points of the surface. A quartic surface having a double conic in the plane at infinity may be put in the form

$$\alpha Q^2 + (\beta x)(x)\, Q + (\psi x)^2 (x)^2 = 0$$

where α is a constant, (βx) an arbitrary plane, $(\psi x)^2$ an arbitrary quadric and (x) the plane at infinity. Q is the known quadric cutting out the double conic in $(x) = 0$. For simplicity let Q contain the four lines joining the four double points. Then $Q \equiv x_0 x_2 + \mu x_1 x_3 = 0$. The surface becomes

$$\alpha (x_0 x_2 + \mu x_1 x_3)^2 + (\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) \times$$
$$(x_0 + x_1 + x_2 + x_3)(x_0 x_2 + \mu x_1 x_3) +$$

$$(x_0+x_1+x_2+x_3)^2(f_{00}x_0^2+f_{11}x_1^2+f_{22}x_2^2+f_{33}x_3^2+ 2f_{01}x_0x_1+2f_{02}x_0x_2+2f_{03}x_0x_3+2f_{12}x_1x_2+ 2f_{13}x_1x_3+2f_{23}x_2x_3)=0$$

Applying the condition that the four reference points are double points on the surface we have the relations

$$f_{00}=f_{11}=f_{22}=f_{33}=f_{01}=f_{03}=f_{12}=f_{23}=0.$$

$$-2f_{02}=\beta_0=\beta_2. \qquad \frac{-2f_{13}}{\mu}=\beta_1=\beta_3.$$

These values of the constants give a general form of the equation with two arbitrary constants, μ and $\frac{\alpha-\beta_0}{2\alpha}\equiv\lambda$. The equation is

$$(1)\ (x_0x_2+\mu x_1x_3)^2+\lambda(x)\{x_0x_2(x_1+x_3)- \mu x_1x_3(x_0+x_2)\}=0.$$

From the restrictions put on Q, we see that this surface contains all the edges of the reference tetrahedron except $\overline{02}$ and $\overline{13}$, and these lines meet the plane $(x)=0$ at points on the double conic.

§1. The Determination of a Set of Correlations.

To determine a correlation by which to test the self-duality of this surface, we may make use of the fact that in a self-dual surface the singularities of the surface in points and the surface in planes are mutually dual. We must then look first for four double tangent planes and a double quadric cone to correspond to the four double points and the double conic which we know are present in a [illegible] cyclide.

To find the double tangent planes, we must first consider the nature of the double points to which they are to correspond. If we write the equation of the surface according to the powers of one of its variables, we see that the tangent at a double point is a quadric cone. The double points are then "conical points".

at which there are an infinite number of tangent planes enveloping a quadric cone. The dual singularity must then be a plane containing an infinity of points of the surface lying on a conic. So the double points lie by pairs on the lines $\overline{13}$ and $\overline{02}$ of the reference tetrahedron. We examine first the relation to the surface of the planes on these lines. Take the planes on $\overline{13}$ which are of the form $k x_0 + x_2 = 0$. The tangent cones at $(0,1,0,0)$ and $(0,0,0,1)$ are respectively

(2). $\mu^2 x_3^2 + \lambda x_0 x_2 - \mu \lambda x_3 (x_0 + x_2) = 0$

(3). $\mu^2 x_1^2 + \lambda x_0 x_2 - \mu \lambda x_1 (x_0 + x_2) = 0$.

The line pairs in which a plane of the pencil cuts these cones coincide for $k = \frac{\lambda - 2 \pm 2\sqrt{1-\lambda}}{\lambda}$. There are then in the pencil two planes given by these values of k which are tangent to both of the tangent cones at these double points;

and tangent to the surface at these points. Moreover the quartic curve in which each of these planes cuts the surface reduces to a repeated conic given by

$$(4)\quad \begin{aligned} &k\,x_0 + x_2 = 0 \\ &\left\{ 2(k\,x_0^2 + \mu\,x_1 x_3) + \lambda(k-1)\,x_0(x_1 + x_3) \right\}^2 = 0. \end{aligned}$$

At all additional points on this conic there is found to be a single tangent plane to the surface, $k x_0 + x_2 = 0$, so that the plane is a trope. Similarly there are among the planes on OZ, $l x_1 + x_3 = 0$, two tropes given by $l = \dfrac{\lambda + 2\mu \pm 2\sqrt{\mu\lambda + \mu^2}}{\lambda}$. We have, then, found four planes which have the properties required for the planes dual to the four double points. These four tropes are

$$(5)\quad \begin{cases} k_1 x_0 + x_2 \equiv \dfrac{\lambda - 2 + 2\sqrt{1-\lambda}}{\lambda}\, x_0 + x_2 = 0. \\ k_2 x_0 + x_2 \equiv \dfrac{\lambda - 2 - 2\sqrt{1-\lambda}}{\lambda}\, x_0 + x_2 = 0 \\ l_1 x_1 + x_3 \equiv \dfrac{\lambda + 2\mu + 2\sqrt{\lambda\mu + \mu^2}}{\lambda}\, x_1 + x_3 = 0 \\ l_2 x_1 + x_3 \equiv \dfrac{\lambda + 2\mu - 2\sqrt{\lambda\mu + \mu^2}}{\lambda}\, x_1 + x_3 = 0 \end{cases}$$

The double conic of the surface is characterized by the fact that at each point the surface has two tangent planes. The dual of the conic must then be a quadric cone, every plane of which meets the surface in two points. Further, the tangent planes of the surface at points of the conic meet the plane $(x) = 0$ in lines tangent to the conic. Hence, dually, the two contacts of each plane of the cone lie on a line passing through a fixed point, the vertex of the cone. If then we consider this as a cone of double tangent lines, we may consider its vertex as a point dually related to the plane of the double conic.

Before finding the equation of this cone, and its vertex, it will be necessary to notice the relation which the points of contact of the elements of such a cone have to the vertex.

Since the elements are double tangent lines of the surface, they meet it in four points given by a binary quartic with two pairs of equal roots. If 0 and ∞ are the parameters of the points of contact on any element, the binary quartic is $x_1^2 y_2^2$. Let the unity be z_1+z_2. The first polar of $x_1^2y_2^2$ as to (z) is $(z_1y_2+z_2x_1)x_1y_2$; the third, $z_1z_2(z_1y_2+x_1z_2)$. These have a common point $(z_1y_2+x_1z_2)$. As this relation is true on every element, it means that the first polar of the unity as to the quartic surface meets the third polar in a series of points which lie on the parabolic curve, that is in a conic. The complete intersection is then a conic and a line. Since the anharmonic ratio of $z_1+z_2=0$, $z_1y_2+z_2x_1=0$, $x_1=0$, $y_2=0$ is -1, we see that the elements of this cone meet the polar plane of the unity in points harmonic with the

unity and the two points of contact on the element.

The equation of the cone is most easily found if we write the equation of the surface in terms of the parameter of the point $x_i + y\eta_i$ in which a line on x_i and η_i meets it. The equation becomes

(6). $p_4' + p_3' y + \frac{1}{2} p_2 y^2 + p_1 y^3 + p y^4 = 0$,

where p = equation of surface.

p_i = i^{th} polar of surface as to a point

$3!\, p_3' = p_3$. $4!\, p_4' = p_4$.

If the line cutting the surface is an element of a cone of double tangent lines, it meets the surface in two points only, and (6) has two pairs of equal roots. We may take 2 as the unity. If for simplicity we set $p_1 = 0$, we know from the preceding paragraph that $p_3 = 0$ also, and the equation reduces to the quadratic

(7) $p_4' + \frac{1}{2} p_2 y^2 + p y^4 = 0$.

The condition for equal roots is now the vanishing of the discriminant, $16p_4'p - p_2^2$. This expression set equal to zero gives an equation in z and x which vanishes when x is a point on a double tangent through it. It must, then, for some special value of z be the equation of the double tangent cone from z.

The vertex of this cone is the point [illegible]. This fact is most easily determined by verifying that for this value of z the discriminant $16p_4'p - p_2^2 \equiv 0$. The p's, found for this point are

$$(8).\begin{cases} p_1 = 2\left[(x\{(\mu Q+\lambda R) + \lambda(\mu+1)\}x_0x_2(x_1+x_3) \right. \\ \qquad \left. - \mu x_1x_3(x_0+x_2)\}\right]. \\ p_2 = 4\left[(\mu+1)(\mu Q + 2\lambda R)\right) \\ \qquad\qquad Q \equiv x_0x_2+\mu x_1x_3. \quad R \equiv x_0x_2 - \mu^2 x_1x_3. \\ p_3' = 2\mu(\mu+1)\{(\mu+\lambda)(x_0+x_2)+\mu(1-\lambda)(x_1+x_3)\} \\ p_4' = \mu^2(\mu+1)^2. \end{cases}$$

substituting in the discriminant these values of p_i we have as the equation of the cone

$$(9).\ 4(\mu+1)(\mu Q+\lambda R)-\{(\mu+\lambda)(x_0+x_2)+\mu(1-\lambda)(x_1+x_3)\}^2=0.$$

A section of this surface by $p_3'=0$ gives

$$(10).\ (\mu Q+\lambda R)\{(X)^2-4\lambda(\mu+1)R\}=0.$$

But a surface on the intersection of $p_1=0$ and $p_3'=0$ is easily seen to be $\mu Q+\lambda R=0$; hence $\mu Q+\lambda R$ is known to be identically zero since the polar plane of $(\mu,1,\mu,1)$ as to the quartic is zero. Therefore the discriminant of (7) vanishes for this point and we see that the cone of double tangent lines, the dual of the double conic in $(X)=0$, is a quadric cone with vertex at $(\mu,1,\mu,1)$. This point is accordingly the dual of $(X)=0$.

We have now four pairs of points and planes associated with [illegible] cyclide, and closely related to each other, — the four double points corresponding to the four tropes, the vertices of

the cone of double tangent lines to the plane at infinity. These are sufficient to determine the correlation which we wish to consider.

Let the correlation be given by the transformation ($u_i = \alpha_{i0} x_0 + \alpha_{i1} x_1 + \alpha_{i2} x_2 + \alpha_{i3} x_0$ and let us choose the case in which $\alpha_{ik} = \alpha_{ki}$. This will give rise to a quadric with respect to which the corresponding point and plane will be pole and polar. In pairing the points and planes a point must not, then, be taken on its corresponding plane. Let the points on $\overline{13}$ and $\overline{02}$ be paired with the planes on $\overline{02}$ and $\overline{13}$ respectively.

	Point	Plane
	$(1, 0, 0, 0)$	$k_1 x_0 + x_2$
(11).	$(0, 1, 0, 0)$	$l_1 x_1 + x_3$
	$(0, 0, 1, 0)$	$k_2 x_0 + x_2$
	$(0, 0, 0, 1)$	$l_2 x_1 + x_3$.

If we regard the equation of each plane as the

result of applying to the corresponding point the transformation $\rho u_i = \sum_k \alpha_{ik} \Psi_k$. $(k=0,1,2,3)$, where $\alpha_{ik} = \alpha_{ki}$, we determine six relations among the coefficients α_{ik}:—

$$\alpha_{03} = \alpha_{01} = \alpha_{12} = \alpha_{23} = 0$$

$$\frac{\alpha_{00}}{\alpha_{22}} = \frac{\alpha_{11}}{\alpha_{33}} = 1$$

The further requirement that by the same transformation $(-1,1,-1,1)$ goes into $(x$ [illegible] three ratios:—

$$\frac{\alpha_{33}}{\alpha_{00}} = \frac{\alpha_{11}}{\alpha_{00}} = \frac{(k+1)\mu l_1}{(1+l_1)k_1}.$$

$$\frac{\alpha_{13}}{\alpha_{00}} = \frac{\mu(1+k_1)}{k_1(1+l_1)}.$$

But such an involutory transformation is a polarity with respect to a quadric $(\alpha x)^2 = 0$, the coefficients of which are the constants of the transformation. Hence we have the quadric

(12). $\Phi_1 \equiv (\alpha x)^2 \equiv k_1(1+l_1)x_0^2 + \mu(1+k_1)l_1 x_1^2 + \mu(1+l_1)x_2^2 + \mu l_1(1+k_1)x_3^2 + 2k_1k_2(1+l_1)x_0x_2$

$+ 2l_1 l_2 \mu(1+k_1) x_1 x_3 = 0$

As k and l have two values connected by the relation $k_1 k_2 = l_1 l_2 = 1$, and so may be combined in four ways, there are four quadrics which meet the conditions and hence four polarities by which certain points of the surface are sent into its planes. The [illegible] [illegible] quadrics are

(13). $Q_2 \equiv k_1(1+l_2)x_0^2 + \mu l_2(1+k_1)x_1^2 + k_1(1+l_2)x_2^2 + \mu l_2(1+k_1)x_3^2 + 2k_1k_2(1+l_2)x_0x_2 + 2l_1l_2\mu(1+k_1)x_1x_3 = 0$

(14). $Q_3 \equiv k_2(1+l_1)x_0^2 + \mu l_1(1+k_2)x_1^2 + k_2(1+l_1)x_2^2 + \mu l_1(1+k_2)x_3^2 + 2k_1k_2(1+l_1)x_0x_2 + 2l_1l_2\mu(1+k_2)x_1x_3 = 0.$

(15). $Q_4 \equiv k_2(1+l_2)x_0^2 + \mu l_2(1+k_2)x_1^2 + k_2(1+l_2)x_2^2 + \mu l_2(1+k_2)x_3^2 + 2k_1k_2(1+l_2)x_0x_2 + 2l_1l_2\mu(1+k_2)x_1x_3 = 0.$

In [illegible] [illegible] there [illegible] a special transformation, polarity as to four quadrics, which

which the point configuration of Kummer's
figure into their closed form.

Moreover the entire surface is completely self-dual under this correlation. The presence of the double points and planes, the double conic and double cone imposes on the surface thirty-four conditions and therefore determines it uniquely. Of these conditions each double point and plane gives one. That the double conic and cone each give ten to these conditions may be seen by writing the equation of the surface according to the powers of one of the variables, x_0. It has then the form $Q + x_0 C + x_0^2(\ ) + x_0^3(\ ) + x_0^4 = 0$, where Q and C are respectively quartic and cubic functions in x_1, x_2, x_3, [illegible]. If we break Q into the square of a quadric cone and C into a plane and the same cone, we have the surface in a form which shows that the intersection of the plane C_2 and the cone Q_1 is

a conic of double points. These are independent of the double conic of the surface (p. 7). To have the quartic cone into the square of a particular cone is 4+5+5 conditions. To break the cubic cone into a plane and the conic quadric cone is 2+5 conditions. From the total of 21 conditions we must subtract the four arising from the condition fixed for convenience. By definition the double conic lies in a fixed plane, so [illegible] three conditions about that plane. But by fixing the plane to be fixed the double both of X and of C at (1.17), and these have each pair of quadric cones Q_1 and Q_2, Q_1 and C_1, coincide by only four conditions, instead of the usual number, eight. We thus counted this point twice and must subtract these from the remaining 16 conditions. Hence it is thirteen conditions on a quartic surface to have a conic of double points. Using the [illegible] space

tion, we see by a similar argument that by imposing thirteen conditions on our expressions that any plane on a fixed point, including an arbitrary one, meets the surface in two points. This was characteristic of the planes enveloping the double curve (p. 7). Hence the surface meets thirty-four conditions. We have then a unique quartic surface which without degenerating admits transformations changing five arbitrary points of the surface into planes of the surface. Hence there must be a [illegible] of every point of the surface, and we have the conclusion that Dupin's Cyclide is a surface self-dual under four correlations.

§2. Study of Covariant Forms.

Before studying these [illegible] further, it is perhaps worth while to examine in some detail the geometry of certain covariant forms closely associated with the surface. Among these forms are the quadrics just found; the developing cone forms [illegible], the tangent cones at the double points, certain points and planes arising in a study of the double conics, and a quartic curve on the cone of double lines.

We will consider first the relations among the quadrics. Q_i Because $k_1 k_2 = l_1 l_2 = 1$, it is possible to write all the quadrics in terms of k_1 and l_1, and in this form it is easily seen that a linear relation exists among them. The equations are

$$(16)\begin{cases} Q_1 \equiv (1+l)(k x_0^2 + k x_2^2 + 2 x_0 x_2) + \mu(1+k)(l x_1^2 + l x_3^2 + 2 x_1 x_3) = 0 \\ l Q_2 \equiv (1+l)(k x_0^2 + k x_2^2 + 2 x_0 x_2) + \mu(1+k)(x_1^2 + x_3^2 + 2 l x_1 x_3) = 0 \\ k Q_3 \equiv (1+l)(x_0^2 + x_2^2 + 2 k x_0 x_2) + \mu(1+k)(l x_1^2 + l x_3^2 + 2 x_1 x_3) = 0 \\ l k Q_4 \equiv (1+l)(x_0^2 + x_2^2 + 2 k x_0 x_2) + \mu(1+k)(x_1^2 + x_3^2 + 2 l x_1 x_3) = 0 \end{cases}$$

If we combine these equations in pairs in all possible ways, we have the following relations:

$$(17).\ \begin{cases} Q_1 - l\,Q_2 = \mu(1+k)(l-1)(x_1-x_3)^2 \\ Q_3 - l\,Q_4 = \frac{\mu}{k}(1+k)(l-1)(x_1-x_3)^2 \end{cases}$$

$$(18).\ \begin{cases} Q_1 - k\,Q_3 = (l+1)(k-1)(x_0-x_2)^2 \\ Q_2 - k\,Q_4 = \frac{1}{l}(l+1)(k-1)(x_0-x_2)^2 \end{cases}$$

For each of these pencils of quadrics the invariant[*] $\phi = 0$.

$$(19).\ \begin{cases} l\,Q_2 - k\,Q_3 = (1+l)(k-1)\{(x_0-x_2)^2 + \mu(1+k)(1-l)(x_1-x_3)^2\} = 0 \\ Q_1 - kl\,Q_4 = (1+l)(k-1)\{(x_0-x_2)^2 + \mu(1+k)(l-1)(x_1-x_3)^2\} = 0 \end{cases}$$

For these two quadrics the invariants[*] Θ and $\Theta' \equiv 0$.

$$(20).\ \begin{cases} l\,Q_2 + k\,Q_3 = (1+l)(1+k)\{(x_0+x_2)^2 + \mu(x_1+x_3)^2\} \\ Q_1 + kl\,Q_4 = (1+l)(1+k)\{(x_0+x_2)^2 + \mu(x_1+x_3)^2\}. \end{cases}$$

From (17) or (19) we see that

$$(21).\ Q_1 - k\,Q_3 - l\,Q_2 + kl\,Q_4 \equiv 0$$

Hence, any one quadric is a linear function of the other three.

From (17) we see further that the pairs Q_1 and Q_2, Q_3 and Q_4 have a repeated conic in common in the plane $x_1 - x_3 = 0$, and are accordingly tangent

along the conic in which they meet it. Equations (18) give the same relation for Q_1 and Q_3, Q_2 and Q_4, which have their [illegible] [illegible] in $x_0-x_2=0$. Each quadric is seen to contain a fixed conic in $x_0-x_2=0$ and in $x_1-x_3=0$. The intersection of these planes thus meets all four quadrics in the same two points; and at these points the quadrics have double tangency. The points are $(\pm\sqrt{-\mu}, 1, \pm\sqrt{-\mu}, 1)$. The common tangent planes at these points are

$(x_0-x_2) \mp \sqrt{-\mu}\,(x_1-x_3)=0$.

From (19) we learn that Q_1 and Q_4, Q_2 and Q_3 meet also in other conics in the planes $x_1+x_3=0$, $x_0+x_2=0$, and accordingly have double contact along the line $\begin{cases} x_1+x_3=0 \\ x_0+x_2=0 \end{cases}$ in each pair. The points of contact of Q_1 and Q_4 are

$\sqrt{-\mu(1+k)(1-k)}, \pm\sqrt{(1+k)(1-k)}, -\sqrt{-\mu(1+k)(1-k)}$,

$\mp\sqrt{(1+k)(1-k)}$;

Of Q_2 and Q_3,

$\sqrt{\mu(1+k)(1-i)}, \pm\sqrt{(1+i)(1-k)}, -\sqrt{\mu(1+k)(1-i)},$
$\mp\sqrt{(1+i)(1-k)}$.

The common tangent planes for the first pair are

$$(x_0-x_2)\sqrt{(1+i)(1-k)} \mp (x_1-x_3)\sqrt{-\mu(1+k)(1-i)}=0;$$

for the second,

$$(x_0-x_2)\sqrt{(1+i)(1-k)} \mp (x_1-x_3)\sqrt{\mu(1+k)(1-i)}=0.$$

Each quadric has three "ring contact" with two others and double contact with a third. At two points on one fixed line all four quadrics have double contact. Also a second [illegible] have double contact by pairs at two sets of points which form a harmonic set.

The enveloping cone to a surface of the fourth order from a point without it is of order twelve—since it consists of tangent lines drawn to the intersection of the surface and its first polar. The first polar of (x) with respect to [illegible] is

$$(22''). \ (x)\left\{(\mu+\lambda)x_0x_2+\mu^2(1-\lambda)x_1x_3\right\}+\lambda(\mu+1)x_0x_2\overline{x_1+x_3}-$$

$\mu x_1 x_3 \overline{x_0+x_2})=0.$

This meets the plane $(x)=0$ at the double conic

$$\begin{cases} x_0x_2+\mu x_1x_3=0 \\ (x)=0 \end{cases}$$

This cubic also contains the double points of the surface and the lines joining them, along which it is tangent to the surface. The complete intersection of the surface with these consists of a cone to the double conic, counted twice, the four planes tangent to the surface along its four lines, and the cone of double lines already found. The planes are

$$(23).\quad 1).\ x_0-\mu x_1=0 \qquad 3).\ x_2-\mu x_1=0$$
$$2).\ x_0-\mu x_3=0. \qquad 4).\ x_2-\mu x_3=0.$$

The cone to the double conic is

$$(24).\ 4(\mu+1)(x_0x_2+\mu x_1x_3)-\mu(x)^2=0.$$

The lines along which the planes (23) are tangent to the surface meet the double cone in four points whose coordinates are respectively

(25). $(0,0,1,-1)$; $(0,1,-1,0)$; $(1,0,0,-1)$; $(1,-1,0,0)$. They meet the cone of double lines in points

(26). $\{0,0,\mu(1-\lambda),-(\mu+\lambda)\}$; $\{0,-(\mu+\lambda),\mu(1-\lambda),0\}$; $\{\mu(1-\lambda),0,0,-(\mu+\lambda)\}$; $\{\mu(1-\lambda),-(\mu+\lambda),0,0\}$.

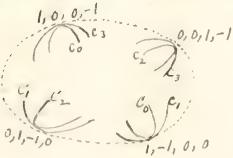
The planes (23) are then tangent to both cones. Hence we see that the enveloping cone from $(\mu,1,\mu,1)$ to the surface consists of two quadric cones and their four common tangent planes.

At the points (25) the double conic is met also by the four tangent cones on the double points of the surface. These cones are

$$(27).\begin{cases} C_0: & x_2^2+\lambda x_1x_2+\lambda x_2x_3-\mu\lambda x_1x_3=0 \\ C_1: & \mu^2x_3^2+\lambda x_0x_2-\mu\lambda x_0x_3-\mu\lambda x_2x_3=0 \\ C_2: & x_0^2+\lambda x_0x_1+\lambda x_0x_3-\mu\lambda x_1x_3=0 \\ C_3: & \mu^2x_1^2+\lambda x_0x_2-\mu\lambda x_0x_1-\mu\lambda x_1x_2=0 \end{cases}$$

Each cone contains two lines of the surface. The pairs of cones with vertices on the diagonals $\overline{02}$, $\overline{13}$ which are touched by the tropes, meet each other

[illegible] points. [illegible] two cones, except three pairs, have one line in common. Along this line they are touched by one of the four tangent planes (23). In (x)=0, then, we have at each of the four points (25) two of the cones tangent to each other and to



the conic. Each cone has double contact with the conic, but single contact only with each of the two cones which it meets.

On page 8 we noticed that each element of the double tetrahedral had on it four points, [illegible] the vertex of the cone and the two points of contact with the Kummer. As there is a reciprocal relation between the points, planes and lines of the [illegible] points and the planes, points and lines [illegible] in planes, [illegible] cone, [illegible]

double conic on which are four harmonic planes. Three must be $(x)=0$, which corresponds to the vertex, and the two tangent planes of the surface at this point, which correspond to the coincidence contact on the element. As the aggregate of fourth points lay on a conic, so here the aggregate of the fourth planes should lie on a cone.

To find this cone we first write the two tangent planes of the surface at a point y of the double conic. They are the factors of the polar quadric of y:

$$(28).\quad 2\{x_0y_2+x_2y_0+\mu(x_1y_3+x_3y_1)\}-\{\lambda(y_0+y_2)\pm\sqrt{\lambda^2(y_0+y_2)^2-4\lambda y_0y_2}\}(x)=0.$$

The fourth harmonic of these and $(x)=0$ is

$$(29).\quad x_0y_2+x_2y_0+\mu(x_1y_3+x_3y_1)-\lambda\frac{y_0+y_2}{2}(x)=0.$$

All such planes pass through $x_0=x_2=\mu+\lambda;\ x_1=x_3=1-\lambda$, which is therefore the vertex of the cone enveloped by the fourth harmonic planes. The cone is

$$(30).\quad 4(\mu+1)(x_0x_2+\mu x_1x_3)-2(x)(x_0y_2+y_0x_2+$$

$\mu\,\overline{x_1 y_3 + x_0 y_1}) + (\lambda^2 + \mu)\,x^2 = 0.$

Equation (28) at once gives information about four important points of the surface. When the expression under the radical is zero the two tangent planes coincide. Solving $\lambda^2(y_0 + y_2)^2 - 4\lambda y_0 y_2 = 0$, we find that a point with coordinates $\frac{y_2}{y_0} = -\frac{\lambda - 2 \pm 2\sqrt{1-\lambda}}{\lambda}$ has such coincident planes. Such a point on the double conic has also $\frac{y_3}{y_1} = -\frac{\lambda + 2\mu \pm 2\sqrt{\mu^2 + \lambda\mu}}{\lambda}$, but these constants are respectively the values of k_i and l_i of the double tangent planes (5) and the ratios become $\frac{y_2}{y_0} = -\frac{k_i}{1}$; $\frac{y_3}{y_1} = -\frac{l_i}{1}$. Each of these four points satisfies two tropes, $(x)=0$, and the cone of double lines, as well as the double conic. They are then the vertices of the quadrilateral cut out by the tropes in $(x)=0$ and inscribed in the double conic at the points where it meets the [illegible]. The peculiarity of these four points, which we call "pinch

points are

(31). $-(l_i-1), (k_i-1), k_i(l_i-1), -l_i(k_i-1)$.

The tangent planes of the surface at these points are

(32). $(l_i+1)(1-\lambda)(k_i x_0+x_2) - (k_i+1)(\mu+\lambda)(l_i x_1+x_3)=0$

The fact that the tropes pass through these points enables us to state the intersections between the singular conics in these planes. As we have already seen, the conics of a pair of tropes on a diagonal line meet in the double points on this line. At each of the "pinch points" two tropes meet which are not on a diagonal, and hence the corresponding singular conics meet also. Each conic accordingly meets one other conic at two double points of the surface and a second at a "pinch point." These intersection points all three seem to have the following properties. They are the end points of the double conic at which the tangent cone is a single tangent plane, and they

are each found on the double conic, the cone of double lines, two tropes and two singular conics.

The duals of the pinch points are planes already found. At each point the single tangent plane of the surface cuts $x_i = 0$ in a line tangent to the double conic. Dually, on the cone of double lines must be an element meeting the surface in two coincident points. The tangent plane on this line meets the surface in four coincident points. Moreover the pinch points are common to two singular conics not lying in tropes on a diagonal. The corresponding planes must then be tangent planes of the cones of the surface at double points not on the same diagonal. Such planes are the four singular planes (23), and the coincident points of contact are the points (25) which are on the conic cut from the cone of double lines by the polar plane of its vertex. The planes of the [illegible] dual to the

[illegible] points [illegible] the [illegible] planes of the surface.

The only space curve on the surface which arises in the preceding discussion is the locus of the points of contact of the cone of tangent lines and the surface. This curve is also on the polar of the surface as to $(\mu,1,\mu,1)$. As the intersection of this cubic surface and the cone [illegible] a sextic surface; but since the further intersection of these two surfaces is a plane conic (p. 8) the locus of contacts is a quartic curve of the "first sort", that is it is cut out by a pencil of quadrics. (Salmon, Geometry of Three Dimensions, 4th Edition, p. 363).

By combining the equations of the surface, the first polar as to $(\mu,1,\mu 1)$ and the cone we obtain a quartic surface which must contain all points common to the three surfaces, and hence the curve. Such a surface is

(31). $$\left[2(\mu+1)(x_0x_2+\mu x_1x_3) - (X)\{(\mu+\lambda)(x_0+x_2) + \mu(1-\lambda)(x_1+x_3)\}\right] \times \left[2(\mu+1)(x_0x_2+\mu x_1x_3) + (X)\{(\mu+\lambda)(x_0+x_2) + \mu(1-\lambda)(x_1+x_3)\}\right] = 0.$$

To decide which of these quadrics cuts out the curve under discussion, we may put the equation of the surface in a form which will show its relation to the cone of double lines. If S is the surface, C_1 the cone of double lines, Q_1 a quadric through the contacts and Q_2 an arbitrary quadric, then $S = Q_1^2 - C_1Q_2$ is such a form. This expresses that S is tangent to C_1 where C_1 meets Q_1. Hence, that one of the two quadrics (31) [illegible] [illegible], the same the square of which subtracted from the surface gives the cone multiplied by some quadric factor. Q_1 is found to be

(32). $$2(\mu+1)(x_0x_2+\mu x_1x_3) - (X)\{(\mu+\lambda)(x_0+x_2) + \mu(1-\lambda)(x_1+x_3)\} = 0. \qquad Q_2 = 4(\mu+1)(X^2).$$

The pencil of quadrics which cut out the

kinetic energy of rotation of the [illegible] of [illegible] [illegible] in the [illegible] [illegible] [illegible] [illegible]tion (34) and (4).

§3. The group of Correlations on the Surface.

The four correlations (§1) and three known collineations which leave the surface unaltered constitute a group, G_8.

C_1.	C_2.	C_3.
$\rho x_0' = x_2$	$\rho x_0' = x_0$	$\rho x_0' = x_2$
$\rho x_1' = x_1$	$\rho x_1' = x_3$	$\rho x_1' = x_3$
$\rho x_2' = x_0$	$\rho x_2' = x_2$	$\rho x_2' = x_0$
$\rho x_3' = x_3$.	$\rho x_3' = x_1$	$\rho x_3' = x_1$.

Forming the polars of a point y as to the quadrics $\varphi_i = 0$, (12)-(15), we have

$$P_1 \equiv (1+l)(ky_0+y_2)x_0 + \mu(1+k)(ly_1+y_3)x_1 + (1+l)(y_0+ky_2)x_2 + \mu(1+k)(y_1+ly_3)x_3.$$

$$lP_2 \equiv (1+l)(ky_0+y_2)x_0 + \mu(1+k)(y_1+ly_3)x_1 + (1+l)(y_0+ky_2)x_2 + \mu(1+k)(ly_1+y_3)x_3.$$

(35).

$$kP_3 \equiv (1+l)(y_0+ky_2)x_0 + \mu(1+k)(ly_1+y_3)x_1 + (1+l)(ky_0+y_2)x_2 + \mu(1+k)(y_1+ly_3)x_3.$$

$$lkP_4 \equiv (1+l)(y_0+ky_2)x_0 + \mu(1+k)(y_1+ly_3)x_1 + (1+l)(ky_0+y_2)x_2 + \mu(1+k)(ly_1+y_3)x_3.$$

If we transform any one of these polarities by any one of the three collineations, the result is the equation of another polar. For example C_1P_1 gives P_3. The same transformation on C_0 gives $C_1C_0 = C_2$. As the result of carrying out these transformations in all possible ways, we have [illegible] from which we learn the following facts:-

1). The product of two like collineations or two like polarities is identity.

2). the product of two unlike collineations gives the third, and of two unlike polarities gives a collineation.

3). Each polarity and each collineation is its own inverse.

4). the products are commutative.

All requirements for a group are thus met and we may state our result in a theorem:

The four polarities P_i and the three collineations C_i, with identity, form a group, G_8.

defined by the following equations

$C_m^2 = 1.$ $\qquad$ $P_i^2 = 1$

$C_m = C_m^{-1}$ $\qquad$ $P_i = P_i^{-1}$

$C_m C_k = C_l$ $\qquad$ $P_i P_j = C_m.$

$m, k, l = 1, 2, 3; \; m \neq k \neq l.$ $\qquad$ $i, j = 1, 2, 3, 4. \quad i \neq j.$

$C_m C_k = C_k C_m.$ $\qquad$ $P_i P_j = P_j . P_i.$

The generating elements are the substitutions [illegible] and C_2 [illegible] [illegible]. The substitutions form the only subgroup. which is a "four-group".

The multiplication table of [illegible] is:

(36).

	1	C_1	C_2	C_3	P_1	P_2	P_3	P_4
1	1	C_1	C_2	C_3	P_1	P_2	P_3	P_4
C_1	C_1	1	C_3	C_2	P_2	P_1	P_4	P_3
C_2	C_2	C_3	1	C_1	P_3	P_4	P_1	P_2
C_3	C_3	C_2	C_1	1	P_4	P_3	P_2	P_1
P_1	P_1	P_2	P_3	P_4	1	C_1	C_2	C_3
P_2	P_2	P_1	P_4	P_3	C_1	1	C_3	C_2
P_3	P_3	P_4	P_1	P_2	C_2	C_3	1	C_1
P_4	P_4	P_3	P_2	P_1	C_3	C_2	C_1	1

Any equation symmetric in x_0, x_2 and x_1, x_3 is invariant under the sub-group of substitutions. Such forms are the equations of the planes $(x_0 \pm x_2) + k(x_1 \pm x_3) = 0$; the quadrics $x_0x_2 + kx_1x_3 = 0$ and $x_0x_1 \pm x_2x_3 = 0$, and the surface itself. Invariant lines are given by

(1). $x_0 = 0$ (2). $x_1 = 0$ (3) $x_0 + x_2 = 0$ (4). $x_0 - x_2 = 0$

$x_2 = 0$ $x_3 = 0$ $x_1 + x_3 = 0$ $x_1 - x_3 = 0$

The first three are invariant in that they are sent into themselves by G_4. The points of the last however are absolutely invariant.

The four lines meet in four points and determine two pairs of opposite generators of $x_0x_1 - x_2x_3 = 0$. The polar as to the quadric F: of a point on one generator of a pair is a plane on the other. We may then say that these lines are also invariant under G_8.

The points arising from any point on the surface by the operations of the group, [illegible]

a set of eight points which constitute a quad-ric. By the collineations C_i a point (y_0, y_1, y_2, y_3) goes into (y_1, y_0, y_3, y_2), (y_3, y_2, y_1, y_0), (y_2, y_3, y_0, y_1). These three form [illegible] point a X set. By the four polarities Q_i the same point goes into the four planes p_i, which [illegible] therefore a four [illegible] point. These we call the set X. By the polarity properties, (p. 33) we see that whichever of the eight points we take as initial point, the same set of eight results from the operations of the group. Any one of the eight thus determines same.

Take the four planes p_1, p_2, p_3, p_4 determined by forming the polars of y as to the quadrics Φ_i. They meet at a point of which coordinates (37). $(-(y_1+y_3), -(y_0+y_2), -(y_1+y_3), -(y_0+y_2))$. The form of the coordinates shows that this is a point on the line $\begin{cases} x_0 - x_2 = 0 \\ x_1 - x_3 = 0 \end{cases}$. Taking [illegible]

points of the set X lie on a plane, p'. The equation of this is

(12). $(x_0+x_2)(y_1+y_3)-(x_1+x_3)(y_0+y_2)=0$.

This is the polar plane of y' as to all four quadrics Q_i and from its form we see that it passes through the fixed line $\begin{cases} x_0+x_2=0 \\ x_1+x_3=0 \end{cases}$.

By the properties of the group, in particular $\tau_i^2=1$, $\tau_i\tau_j=\tau_k$, the relation between the set of points X and their polar planes, p_k, is in all respects the same as that between the points X and the planes p_i. If we start with a point X the polarities give four planes Q_i tangent to the surface at points X. These planes intersect then also meet at a point, x', on the line $\begin{cases} x_0-x_2=0 \\ x_1-x_3=0 \end{cases}$, and the four points X lie in a plane which is the polar of x' as to Q_i. The coordinates of x' are

(13). $x_0'-x_2'=[2(y_0+y_2)(y_0y_2+y_1y_3)+$ [illegible] $]$ [illegible]

$x_1' = x_3' = -[2u(y_1+y_3)(y_0y_2+uy_1y_3) + 2\lambda\{y_0y_2(y_1+y_3) - uy_1y_3(y_0+y_2)\} + \lambda(y_1+y_3)[2y_0y_2 - u(y_0+y_2)(y_1+y_3)]]$

The polar plane of x' as to G is a plane on $\begin{cases} x_0+x_2=0 \\ x_1+x_3=0 \end{cases}$ The eight points ~~then~~ thus determine a quadric, [illegible], these two planes on the line $\begin{cases} x_0+x_2=0 \\ x_1+x_3=0. \end{cases}$

The pairs of points arising thus on $\begin{cases} x_0-x_2=0 \\ x_1-x_3=0 \end{cases}$ from every point on the surface form an involution along the line, the double points of which are the two common points of the four quadrics in consecutive contact (p. 20). The corresponding pairs of planes on $\begin{cases} x_0+x_2=0 \\ x_1+x_3=0 \end{cases}$ also form an involution, the double planes of which are the two common tangent planes of the four quadrics at their points of contact.

We have then the conclusion that the eight points and eight planes of the [illegible] arising from the operation of the group, on any point of the complex [illegible]

-fours. Each set of four respective points is a point on one fixed line of the group, while each set of four points lies in a plane passing through a second fixed line. The intersection of the plane tangent at one set of four points is the polar point as to the quadrics of the plane containing the four other points.

This group of correlations for rendering the configuration self-dual is the only such group. If a further group existed it would be formed by adjoining some new collineations from the known correlations. Applying the general collineation $\rho x_i' = \sum_k a_{ik} x_k$ to the equation of the surface with the condition that the surface remain unaltered, we find that the only solutions possible are the four transformations given by the substitution factors of 1, -1, i, -i. The correlation group is therefore unique.

§4. Mapping of Surface on an Arbitrary Plane.

By the principle of mapping it is possible to give a proof [illegible] of the theorem that the correlations found in §1 do in fact send every point of Dupin's Cyclide into the plane. To do this we must first show that the surface may be mapped by [illegible] on [illegible] point upon plane.

A surface may be mapped only when it is rational. That Dupin's Cyclide is a rational surface may be seen by considering the intersections with the surface of a line which meets the double conic and the line of the surface. Three of its four intersections are thus known; the fourth is variable. If the cutting line also meets some arbitrary plane, this point and the variable intersection with the surface are always in 1-1

correspondence. The surface then must be that the rationality and may be mapped.

In mapping a surface of given order, the curves which the plane sections must pass through a sufficient number of base points so that the free intersection of any two shall be equal in number to the order of the surface. To map a quartic surface, therefore, by means of plane cubics, we must have five base points. This puts five conditions on each cubic and gives in all ∞^4 cubics. From the ∞^3 planes of the four linear functions of the point coordinates planes which shall be mapped by ∞^3 cubics on the five base points, which are linear functions of the four fundamental cubics. All such curves must be mapped under the further conditions, that the plane sections of the surface are mapped by a linear

system of cubics in the plane.

Let the sixth condition which must be met by these cubics be apolarity to a general line cubic, $(\alpha\xi)^3 = 0$. We may then proceed to derive a Cremona transformation which will map [illegible] [illegible] on a plane. We must first however take account of the fact that the surface has four double points. This [illegible] the plane representation of the cubic surface in general it is assumed that no three points lie on a straight line, but when the surface has a double point, this restriction is removed. This is because each of the net of planes which may be passed through a double point on the surface cuts out a rational section and hence is mapped by a rational cubic. Thus are these images the ∞^3 cubics on [illegible] point and $(\alpha\xi)^3$ [illegible]

rational [illegible]. Two such [illegible] have one

[illegible] intersecting [illegible] a [illegible] three points

and a net of conics on them the corresponding

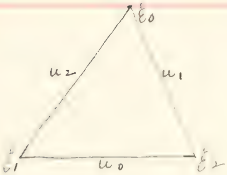
is the polar conic of the line [illegible] $(a\xi)$;

2) a net of [illegible] rational curve [illegible] the

double point at a free point.

Two such in general [illegible] different positions of three points on a line, as in case 1, may be made. Case 2 arises when two of the points on a line approach each other. Some of the conics through these points, a net will here have a double point. Two such nets are possible. We have these [illegible] peculiarities due to the presence of four double points on the surface if we take the five base points in two [illegible] pairs on two lines meeting at the fifth point.

Let ~~three~~ of these points be the vertices of the reference triangle in the plane, and



Let the coincident point be ξ_1 and a point on u_2 consecutive to it; ξ_2 and a consecutive point on $u_1=0$

A cubic curve through these points is :: $\xi_0^3, \xi_1^3, \xi_2^3, \xi_0\xi_1^2, \xi_0\xi_2^2$ and is of the form

$$(.40). \quad (\alpha u)^3 \equiv \alpha_{001}u_0^2u_1 + \alpha_{002}u_0^2u_2 + \alpha_{112}u_1^2u_2 + \alpha_{122}u_1u_2^2 + \alpha_{012}u_0u_1u_2 = 0$$

If $(\alpha u)^3=0$ is also :: $(\alpha\xi)^3=0$ a linear relation exists among the coefficients and (40) becomes

$$(41). \quad (\alpha u)^3 \equiv \alpha_{001}u_0^2u_1 + \alpha_{002}u_0^2u_2 + \alpha_{112}u_1^2u_2 + \alpha_{122}u_1u_2^2 + (a.\alpha_{001} + b.\alpha_{002} + c.\alpha_{112} + d.\alpha_{122})u_0u_1u_2 = 0$$

Making the first four coefficients zero in turn we have four cubics satisfying the imposed conditions

$$(42). \quad \begin{aligned} f_0 &\equiv \beta_0u_0^2u_1 + u_0u_1u_2 = p\,x_0 \\ f_1 &\equiv \beta_1u_0^2u_2 + u_0u_1u_2 = q\,x_1 \\ f_2 &\equiv \beta_2u_1u_2^2 + u_0u_1u_2 = p\,x_2 \\ f_3 &\equiv \beta_3u_1^2u_2 + u_0u_1u_2 = q\,x_3 \end{aligned}$$

To show that this transformation does [illegible]

plane into the surface or are considering m

may eliminate to form these four equations.

If we [illegible] λ and f_0, f_1 and f_2 and

eliminate u_0, u_1, u_2 from the resulting

parametric we have the equation of Dupin's

Cyclide. To make the form identical with (1)

we may set $q = \mu p$; $\beta_0\beta_2 - 1 = -p$; $\beta_1\beta_3 - 1 = -q$;

$p = -\frac{1}{\lambda}$. The transformation then takes the

form

$$
\begin{aligned}
f_0 &\equiv x_0 = \frac{\lambda-1}{\lambda\beta_2}\, u_0^2 u_1 + u_0 u_1 u_2 \\
(43).\quad f_1 &\equiv \mu x_1 = \frac{\mu+\lambda}{\lambda\beta_3}\, u_0^2 u_2 + u_0 u_1 u_2 \\
f_2 &\equiv x_2 = \beta_2 u_1 u_2^2 + u_0 u_1 u_2 \\
f_3 &\equiv \mu x_3 = \beta_3 u_1^2 u_2 + u_0 u_1 u_2 .
\end{aligned}
$$

We are now able to verify algebraically the fact that Dupin's Cyclide is self-dual under the transformation of §11. Since the point [illegible] of a tangent plane is a tangent point on the section of the surface, each component plane [illegible] that the [illegible] surface [illegible]

curve which maps into a rational cubic on this point. The [illegible] mapping of tangent planes [illegible] nodal rational cubics. Correspondingly [illegible] the tangent [illegible] plane [illegible] of the quadrics [illegible] tangent plane [illegible], the locus of the section [illegible] cubic [illegible] a tangent plane at the point [illegible] of y.

$$P_1(x) \equiv (1+l)(ky_0+y_2)x_0 + \mu(1+k)(ly_1+y_3)x_1 + (1+l)(y_0+ky_2)x_2 + \mu(1+k)(y_1+ly_3)x_3 = 0.$$

$$\begin{aligned} P_1(f) \equiv\ & (1+l)(ky_0+y_2)\left\{\frac{\lambda-1}{\lambda\beta_2}u_0^2u_1 + u_0u_1u_2\right\} + \\ & \mu(1+k)(ly_1+y_3)\left\{\frac{\mu+\lambda}{\mu\lambda\beta_3}u_0^2u_2 + \frac{u_0u_1u_2}{\mu}\right\} + \\ & (1+l)(y_0+ky_2)\left\{\beta_2u_1u_2^2 + u_0u_1u_2\right\} + \\ & \mu(1+k)(y_1+ly_3)\left\{\frac{\beta_3}{\mu}u_1^2u_2 + \frac{u_0u_1u_2}{\mu}\right\} = 0. \end{aligned}$$

This cubic is of the form, in Salmon's notation (Higher Plane Curves, 3rd edition, p. 194),

$$(44).\quad a_1u_2^2u_1 + u_2(b_2u_1^2 + b_0u_0^2 + 2b_1u_0u_1) + c_1u_0^2u_1 = 0$$

$a_1 \equiv \beta_2(1+l)(y_0+ky_2)$. $b_2 \equiv \beta_3(1+k)(y_1+ly_3)$.

$b_0 \equiv \frac{\mu+\lambda}{\lambda\beta_3} - (1+k)(l\gamma_1+\gamma_3)$. $c_1 \equiv \frac{\lambda-1}{\lambda\beta_2}(1+l)(k\gamma_0+\gamma_2)$.

$3b_1 \equiv (1+[illegible])(1+k)([illegible])$.

If this cubic is rational its discriminant will vanish. The discriminant is

(45). $[illegible] = c_1 c_1 b_0 b_2 \left[b_1^4 - 2b_1^2(b_0 b_2 + c_1 c_1) + (b_0 b_2 - c_1 c_1)^2 \right]$.

Substituting the values of the coefficients and using the relations* which exist between k, l, λ, μ this becomes

---

*

$l_1 = \left(\frac{\sqrt{\lambda+\mu} + \sqrt{\mu}}{\sqrt{\lambda}}\right)$. $l_2 = \left(\frac{\sqrt{\lambda+\mu} - \sqrt{\mu}}{\sqrt{\lambda}}\right)$

$k_1 = -\left(\frac{\sqrt{1-\lambda} + \sqrt{1}}{\sqrt{\lambda}}\right)$. $k_2 = -\left(\frac{\sqrt{1-\lambda} - \sqrt{1}}{\sqrt{\lambda}}\right)$.

$(k-1)^2 = -\frac{4k}{\lambda}$. $(l-1)^2 = \frac{4\mu l}{\lambda}$

$(k+1)^2 = -\frac{4(1-\lambda)k}{\lambda}$ $(l+1)^2 = \frac{4}{\lambda}(\mu+\lambda)l$

$k^2+1 = 2\frac{k(\lambda-2)}{\lambda}$ $l^2+1 = \frac{2l}{\lambda}(2\mu+\lambda)$

(46). $(y_0+ky_2)^2(ky_0+y_2)^2(y_1+ly_3)^2(ly_1+y_3)^2 \times$

$\left[(y_0y_2+\mu y_1y_3)^2+\lambda(y_{\ldots}) \{ y_0y_2\ [illegible] - \mu y_1y_3\ [illegible] \}\right].$

This expresses that the discriminant vanishes identically when 1) y is a point on a singular conic and when 2) y is any point on the singular surface. As the same result is obtained by [illegible] the pole [illegible] of y [illegible] the two planes generating the correlation is perfectly general, we may again reach the conclusion that [illegible] in any-[illegible] [illegible] position.

Before continuing the discussion of the mapping of the surface, we may note here that the discriminant [illegible] the [illegible] intersection of a tangent plane [illegible] the corresponding [illegible] of the surface. Substituting for y in the discriminant its coordinates as the pole of a plane t_i we have the equation of the surface in [illegible] y, the coordinates of the pole.

$$y_0 = \mu(l-1)(k\eta_0 - \eta_2). \quad y_1 = (k-1)(l\eta_1 - \eta_3).$$

$$y_2 = \mu(l-1)(k\eta_2 - \eta_0). \quad y_3 = (k-1)(l\eta_3 - \eta_1)$$

a simplified form of the surface in planes

$$\left[\mu^2(4\eta_0\eta_2 + \lambda\,\overline{\eta_0-\eta_2}^2) + (4\mu\eta_1\eta_3 - \lambda\,\overline{\eta_1-\eta_3}^2)\right]^2$$
$$+ 4\mu\lambda(\mu\,\overline{\eta_0+\eta_2} + \overline{\eta_1+\eta_3})\left[\mu(\eta_1+\eta_3)\{4\eta_0\eta_2 + \lambda(\eta_0-\eta_2)^2\} - (\eta_0+\eta_2)\{4\mu\eta_1\eta_3 - \lambda(\eta_1-\eta_3)^2\}\right] = 0.$$

The sections of the surface by a general plane is a quintic with six double points. It maps as we have seen into a general cubic on one vertex of the reference triangle and tangent to the sides at the other vertices. It is however of most interest to map, by the transformation (42) all planes which cut the surface in rational sections. Of these there are five distinct sets.

1) Four nets of planes on the double points of the surface. The sections are proper quintic curves with three double points.

2) Four pencils of planes on the lines of the [illegible]-

face. Each section is a rational cubic and the line.

3). Two pencils of planes on the diagonal lines of the tetrahedron. The sections are two conics. The singular planes occur under either 2), 5), and give sections with four double points; or two lines and a conic.

4). ∞' planes tangent to the cone of double lines. The sections are two conics.

The singular planes occur under 2) or 5). The sections have six double points and break into a line counted twice and a conic.

5). ∞² simple tangent planes. The sections are rational quartics with three double points.

We can most directly obtain the images of the double points and lines of the surface by considering the [illegible] equations of the planes through them. The images of the planes through the double point [illegible].

η_0: plane, — $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

Map, — $\left[a_1 \frac{\mu+\lambda}{\mu\lambda\beta_3} u_0^2 + a_2\beta_2 u_1 u_2 + a_3 \frac{\beta_3}{\mu} u_1^2 + \left(\frac{a_1}{\lambda} + a_2 + a_3\right) u_0 u_1\right] u_2 = 0$.

The section maps into $u_2 = 0$ and a conic tangent to $u_1 = 0$ at ξ_2.

η_1: plane, — $a_0x_0 + a_2x_2 + a_3x_3 = 0$.

Map, — of same form as preceding.

The section maps into $u_1 = 0$ and a conic tangent to $u_2 = 0$ at ξ_1.

η_2: plane, — $a_0x_0 + a_1x_1 + a_3x_3 = 0$

Map, — $a_0 \frac{\lambda-1}{\lambda\beta_2} u_0^2 u_1 + a_1 \frac{\mu+\lambda}{\mu\lambda\beta_3} u_0^2 u_2 + a_3 \frac{\beta_3}{\mu} u_1^2 u_2 + \left(a_0 + \frac{a_1}{\mu} + \frac{a_3}{\mu}\right) u_0 u_1 u_2 = 0$.

The section maps into a cubic on the pin-point, tangent to $u_2 = 0$ at ξ_1 and having a double point at ξ_2.

η_3: plane, — $a_0x_0 + a_1x_1 + a_2x_2 = 0$

Map, — of form similar to preceding.

The section maps into a cubic on pin one-point having a double point at ξ_1.

From that part of the map which is common to all planes of a net, apart from the f[illegible] points, [illegible] the maps of the double points.

η_0 maps into u_2. η_2 maps into ξ_2

η_1 " " u_1. η_3 " " ξ_1.

The following are the maps of planes on a line of the surface.

$\overline{\eta_0\eta_1}$: plane, — $a_2x_2+a_3x_3=0$

(47). Map, — $u_1u_2[\mu a_2\beta_2u_2+a_3\beta_3u_1+$

$\mu(a_2+a_3)u_0]=0$

The section maps into three lines.

$\overline{\eta_2\eta_3}$: plane, — $a_0x_0+a_1x_1=0$

(48). Map, — $u_0[\mu(\lambda-1)a_0\beta_3u_0u_1+$

$\lambda(\mu a_0+a_1)\beta_2\beta_3u_1u_2+a_1\beta_2(\mu+\lambda)u_0u_2]=0$

Section maps into u_0 and a c[illegible]

t[illegible] p[illegible].

$\overline{\eta_1\eta_2}$: plane, — $a_0x_0+a_3x_3=0$.

(49). Map, — $u_1[\mu a_0(\lambda-1)u_0^2+a_3\beta_2\beta_3\lambda u_1u_2$

$$+\lambda\beta_2(\mu a_0+a_3)u_0u_2]=0$$

Section mapped into u_1 and conic on ξ_1 and ξ_2 tangent to $u_2=0$ at ξ_1.

$\overline{\eta_0\eta_2}$; plane, $-a_1x_1+a_2x_2=0$

(50). Map, $-\ u_2[a_1(\mu+\lambda)u_0^2+a_2\mu\lambda\beta_2\beta_3u_1u_2$
$+\lambda\beta_3(a_1+\mu a_2)u_0u_1]=0.$

Section mapped into u_2 and conic on ξ_1 and ξ_2 tangent to $u_1=0$ at ξ_2.

The maps of the edges of the tetrahedron are then

$\overline{\eta_0\eta_1}$ — u_1u_2 $\quad$ $\eta_1\eta_2$ — u_1
$\overline{\eta_2\eta_3}$ $\quad u_0$ $\quad$ $\eta_0\eta_3$ — u_2.

Special members of the above families are the singular planes.

Plane.	Map equation.
$X_2-\mu X_3=0$	$u_1u_2[\beta_2u_2-\beta_3u_1]=0$
(51). $X_0-\mu X_1=0$	$u_0^2[(1-\lambda)\beta_3u_1+\beta_2(\mu+\lambda)u_2]=0$
$X_0-\mu X_3=0$	$u_1[(\lambda-1)u_0^2-\beta_3u_1u_2]=0$
$X_2-\mu X_1=0$	$u_2[(\mu+\lambda)u_0^2-\lambda\beta_2\beta_3u_1u_2]=0$

In connection with the mapping of plane sections of the surface, the map of the double conic in (x) = 0 is of interest. This is

$$(52).\quad C \equiv \frac{\lambda-1}{\lambda\rho_2} u_0^2 u_1 + \frac{\mu+\lambda}{\mu\lambda\rho_3} u_0^2 u_2 + \rho_2 u_1 u_2^2 + \frac{\rho_3}{\mu} u_1^2 u_2 + 2\frac{\mu+1}{\mu} u_0 u_1 u_2 = 0.$$

This curve is peculiar in that it is in 1-2 correspondence with a cubic in the mapping plane. The two points of the conic in every plane section appear in the mapping plane as four common points of the maps of the section and the conic. It is shown by Clebsch (Borchardt, Journal für Mathematik, v. 69, s. 147) the pairs of points which map the single points of the double conic lie on a pencil of lines with vertex in the map of the double conic.

This is seen in the present case by considering the maps of planes through the lines of the surface. Every such section

the line and a rational cubic on the intersection of the line and double conic, with a double point elsewhere on the conic. In the case of planes on $\eta\bar{\eta}_1$ (47), its intersections with C of the variable line must map the variable double point in the pencil of cubics. The third intersection and a fourth point must map the fixed intersection of line and cubic. Hence the variable line goes through a fixed point, P, on C and forms a pencil. The tangent to C at P has a residual point P′, which with P maps the intersection of line, cubic and conic. This tangent at P is the map of the plane of the pencil which touches the surface along the line, or the singular plane $x_2 - \mu x_3 = 0$. P has coordinates $(-\mu^2 a_0, b_2, a_2)$; $P' \equiv (1, 0, 0)$.

The discussion of the planes through the three other lines gives the same results

from a slightly different point of view. Consider the planes on $\overline{\eta_2\eta_3}$ (48). The map equation is $u_0 = 0$ and a conic on three reference points. The map of the singular plane in the pencil is $u_0^2\{(1-\lambda)\rho_3 u_1 + \rho_2(\mu+\lambda)u_2\} = 0$. Now $u_0 = 0$ meets C in ξ_1, ξ_2 and $(0, -\mu\rho_2, \rho_3)$. The line meets C in $\{\lambda\rho_2\rho_3, -\rho_2(\mu+\lambda), (1-\lambda)\rho_3\}$ and here touches C. ξ_3 is the residual point. Of these points all but $\{\lambda\rho_2\rho_3, -\rho_2(\mu+\lambda), \rho_3(1-\lambda)\}$ are constant and $(0, -\mu\rho_2, \rho_3)$ are on all conics (48). These however are found only on the conic mapping the singular plane and hence map its contact with the double conic. The cubics in planes on $\overline{\eta_2\eta_3}$ are then mapped by a pencil of conics on four points of C. The two remaining intersections of each conic map the double point of the corresponding cubic, and these, by the symmetry of the plane cubic, lie on a line through a fixed point of C.

This is readily seen to be F, and so are again led to the fact that the pairs of points mapping the points of the double conic determine a pencil of lines through a point on C.

Similarly the cubics in the planes on $\overline{\eta_1\eta_2}$ and $\overline{\eta_0\eta_3}$ map into conics on four points of C. For $\overline{\eta_1\eta_2}$ these conics pass through $\{-\lambda\beta_2\beta_3, \lambda\beta_2, (\lambda-1)\beta_3\}$ and ξ_2, and are tangent to C at ξ_1. The map of the cubic in the singular plane of the pencil is tangent to C at $\{-\lambda\beta_2\beta_3, \lambda\beta_2, (\lambda-1)\beta_3\}$ and at ξ_2. Hence these points map the intersection of $\overline{\eta_1\eta_2}$ and the double conic. The maps of cubics in planes on $\overline{\eta_0\eta_3}$ pass through $\{-\lambda\beta_2\beta_3, (\mu+\lambda)\beta_2, \lambda\beta_3\}$ and ξ_1 and are tangent to C at ξ_2. The first pair of conics, then, maps the intersection of $\overline{\eta_0\eta_3}$ and the double conic. In both cases it is easily verified that conics of the pencils cut out pairs of points on C which lie on the plane pencil P.

We have thus verified for this special surface the result stated by Cayley for the general quartic surface with a double conic. Thus on the lines of the surface are [illegible] mapped by 1) the map of the line; 2) a pencil of curves with four points on the map of the double conic, C. The variable points of intersection of the curves and C map the points of the double conic in 2-1 correspondence, the point pairs of which form a central involution with center on C. We have also shown that when the variable points, coinciding with the fixed points, of the [illegible] of contact between curves of the pencil and C, then these curves map the tangent planes of the pencil. The two intersections of these curves and C, not belonging to the double conic, map the point of contact.

The map equations of planes on the diagonals

of the reference tetrahedron are

$\overline{\eta_1\eta_3}$. plane, — $a_0x_0+a_2x_2=0$.

map, — $u_1\{a_0(\lambda-1)u_0^2+u_0u_2\beta_2(a_0+a_2)\lambda$
$+\lambda\beta_2^2a_2u_2^2\}=0$.

$\overline{\eta_0\eta_2}$. plane, — $a_1x_1+a_3x_3=0$

map, — $u_2\{a_1(\mu+\lambda)u_0^2+\lambda a_3\beta_3^2u_1^2$
$+\lambda\beta_3(a_1+\mu a_3)u_0u_1\}=0$.

The first is $u_1=0$ and two lines on ξ_1; the second is $u_2=0$ and two lines on ξ_2. Special cases of these planes [illegible]

Plane.	Map Equation
(53) $k_ix_0+x_2=0$	$u_1[(k_i+1)u_0+2\beta_2u_2]^2=0$
$l_ix_1+x_3=0$	$u_2[(l_i+1)u_0+2\beta_3u_1]^2=0$.

The points where the loops meet the double conic are mapped by the coincident point case in which the four tangents to C from P meet the conic. Writing the coordinates of these points on Σ we find that the [illegible] of the [illegible] pass through these of some point on the surface.

pass in pairs through the "[illegible] points." The four points are

$$\text{(54).}\quad \begin{array}{ll} u_0 = -2\beta_2\beta_3 & \Big\} \quad u_0 = -2\beta_2\beta_3 l_i k_i \\ u_1 = \beta_2 (l_i+1) & \Big\} \quad u_1 = \beta_2(l_i+1)k_i \\ u_2 = \beta_3(k_i+1) & \Big| \quad u_2 = \beta_3 l_i(k_i+1) \end{array}$$

$i = 1, 2.$

A tangent plane to the cone of double lines is of the form

$$2(\mu+1)\{(\mu+1)(y_0x_2+x_0y_2)+\mu^2(1-\lambda)(x_1y_3+y_1x_3)\}$$
$$-\{(\mu+\lambda)(x_0+x_2)+\mu(1-\lambda)(x_1+x_3)\}\{(\mu+\lambda)(y_0+y_2)+$$
$$\mu(1-\lambda)(y_1+y_3)\}=0.$$

The map of this section is

$$\text{(55).}\quad \begin{array}{l} \beta_3(\mu+\lambda)(\lambda-1)\{2y_2-\lambda(y_0+y_2)\}u_0^2u_1+ \\ \beta_2(\mu+\lambda)(\lambda-1)\{\lambda(y_0+y_2)-2\mu y_3\}u_0^2u_2+ \\ \beta_2^2\beta_3\lambda(\mu+\lambda)\{2y_0-\lambda(y_0+y_2)\}u_1u_2^2+ \\ \beta_2\beta_3^2\lambda(\lambda-1)\{\lambda(y_0+y_2)-2\mu y_1\}u_1^2u_2=0. \end{array}$$

This is a cubic for which the invariants S and T (Salmon) are both zero, and accordingly it has a cusp. Hence the two conics in

which these planes cut the surface are mapped into a cubic with a cusp.

The quartic curve which is the locus of contacts of elements of the double cone is cut out by a quadric

$2(\mu+1)(x_0x_2+x_1x_3) - \lambda\{(\mu+\lambda)(x_1+x_2)+(1-\lambda)x(x_1+x_3)\}=0.$

Substituting $x_i = f_i$ in turn the map equation of the curve. This falls into two parts, one of which is the map of the double conic, the other

$$(56)\quad \beta_3(\mu+\lambda)\{(\lambda-1)u_0^2u_1+\lambda\beta_2^2u_1u_2^2\}+$$
$$\beta_2(1-\lambda)\{(\mu+\lambda)u_0^2u_2+\lambda\beta_3^2u_1^2u_2\}=0$$

This shows that the sextic curve in which the quadric and quartic surfaces meet consists of the double conic, counting as a quartic curve, and the quartic on the double cone. The cubic (56) is in form similar to the map equations of plane sections, but it differs from these in that there is not a [illegible]

existing between it and the four fundamental cubics.

§5. The Group of Cremona Transformations in the Mapping Plane

We have seen that the transformation of polarity as to the fundamental quadrics Φ_i gives for any point y on the surface a plane tangent to the surface. The point of contact, x, is a double point of the section. The map equation of this plane is a rational cubic with double point say at z. Since y is a point of the surface it may be mapped into a point u of the mapping plane. So for a pair of points y and x arising by a correlation on the surface, we have a pair u and z in the plane. We now proceed to obtain a transformation which applied to u, the map of any point y, gives the point z corresponding to x.

The coordinates of the double point z in a curve mapping a tangent plane of the

surface may be found by means of the coefficient

$$(57)\quad f_0(J_{123})_{u=z} + f_1(J_{023})_{u=z} + f_2(J_{013})_{u=z} + f_3(J_{012})_{u=z} = 0$$

The quantities f are the four cubics (43), the quantities J are here the Jacobians of three of the cubics. Since the Jacobian of a net passes through the double points of the net, a curve given by (57) has, for a definite value of z, a double point at z. Let us assume this curve identical with the equation of a tangent plane, and equate coefficients of terms in u. We have then four equations between y and z. But since y may be expressed as a function of z, we have equations in z in terms of u. Combining, we have, corresponding to the tangent plane through a point z on the surface, four sets of relations between z and u.

Π_1 : $z_0 = -\sqrt{\lambda}\,\beta_2\beta_3\{(k+1)(l+1)u_0^2 + 2l(k+1)\beta_3 u_0u_1$

$+ 2k(l+1)\beta_2 u_0u_2 + 4kl\beta_2\beta_3 u_1u_2.$

$z_1 = -\sqrt{(\lambda+\mu)l}\,.\,\beta_2\{(l+1)(k+1)u_0^2 + 2k(l+1)\beta_2 u_0u_2$

$+ 2l(k+1)\beta_3 u_0u_1 + 4kl\beta_2\beta_3 u_1u_2$

$z_2 = -\sqrt{(\lambda-1)k}\,.\,\beta_3\{(k+1)(l+1)u_0^2 + 2l(k+1)\beta_3 u_0u_1$

$+ 2(l+1)\beta_2 u_0u_2 + 4\beta_2\beta_3 l u_1u_2.$

Π_2 : $z_0 = +\sqrt{l\lambda}\,\beta_2\beta_3\{(k+1)(l+1)u_0^2 + 2k(l+1)\beta_2 u_0u_2$

$+ 2(k+1)\beta_3 u_0u_1 + 4k\beta_2\beta_3 u_1u_2.$

$z_1 = -\sqrt{\lambda+\mu}\,.\,\beta_2\{(k+1)(l+1)u_0^2 + 2k(l+1)\beta_2 u_0u_2$

$+ 2l(k+1)\beta_3 u_0u_1 + 4kl\beta_2\beta_3 u_1u_2$

$z_2 = -\sqrt{kl(\lambda-1)}\,.\,\beta_3\{(k+1)(l+1)u_0^2 + 2(l+1)\beta_2 u_0u_2$

$+ 2(k+1)\beta_3 u_0u_1 + 4\beta_2\beta_3 u_1u_2.$

Π_3 : $z_0 = +\sqrt{\lambda k}\,\beta_2\beta_3\{(k+1)(l+1)u_0^2 + 2\beta_2(l+1)u_0u_2$

$+ 2l(k+1)\beta_3 u_0u_1 + 4l\beta_2\beta_3 u_1u_2.$

$z_1 = -\sqrt{kl(\lambda+\mu)}\,.\,\beta_2\{(k+1)(l+1)u_0^2 + 2(l+1)\beta_2 u_0u_2$

$+ 2(k+1)\beta_3 u_0u_1 + 4\beta_2\beta_3 u_1u_2.$

$z_2 = -\sqrt{\lambda-1}\,.\,\beta_3\{(k+1)(l+1)u_0^2 + 2k(l+1)\beta_2 u_0u_2$

$+ 2l(k+1)\beta_3 u_0u_1 + 4kl\beta_2\beta_3 u_1u_2.$

$$\Pi_4:\ z_0 = +\sqrt{\lambda k l}\,\beta_2\beta_3\{(k+1)(l+1)u_0^2 + 2(l+1)\beta_2 u_0u_2$$
$$+ 2(k+1)\beta_3 u_0u_1 + 4\beta_2\beta_3 u_1u_2\}.$$
$$z_1 = -\sqrt{k(\lambda+1)}\,\beta_2\{(k+1)(l+1)u_0^2 + 2\beta_2(l+1)u_0u_2$$
$$+ 2l(k+1)\beta_3 u_0u_1 + 4l\beta_2\beta_3 u_1u_2.$$
$$z_2 = -\sqrt{l(\lambda-1)}\,\beta_3\{(k+1)(l+1)u_0^2 + 2k(l+1)\beta_2 u_0u_2$$
$$+ 2(k+1)\beta_3 u_0u_1 + 4k\beta_2\beta_3 u_1u_2.$$

The inverse transformation, giving u in terms of z, may be written by interchanging u and z in Π_i. We have thus birational transformations which set up a 1-1 correspondence between points u and z. Hence this is a Cremona transformation, and also involutory.

Such a transformation sends the ∞^2 lines of the plane into a net of conics on the three singular points of the transformation. Each quadratic expression for z factors into the maps of two tropes on the surface, and from its form is seen to pass

through ξ_1 and ξ_2. Hence the singular points are easily found to be in each case ξ_1, ξ_2 and the map of our pinch point. The points associated with the transformations are respectively

$$\Pi_1 : \{-2\beta_2\beta_3 k_1,\ \beta_2 k_1(l_1+1),\ \beta_3(k_1+1)\}.$$

$$(58).\ \Pi_2 : \{-2\beta_2\beta_3 k_2,\ \beta_2(l_1+1)k_2,\ \beta_3(k_2+1)\}.$$

$$\Pi_3 : \{-2\beta_2\beta_3 k_1,\ \beta_2 k_1(l_2+1),\ \beta_3(k_1+1)\}.$$

$$\Pi_4 : \{-2\beta_2\beta_3 k_2,\ \beta_2 k_2(l_2+1),\ \beta_3(k_2+1)\}.$$

The singular lines of each transformation, given by the Jacobian of the three quadratics in u, are $u_0 = 0$ and the two last equations of the triple which have in common the same point.

(59). Four points of the plane are fixed under all the transformations Π. Setting $u_i = z_i$ in every equation we have

$$u_0 = z_0 = \pm \beta_2\beta_3\sqrt{\lambda}$$

$$u_1 = z_1 = \pm \beta_2\sqrt{\lambda + \kappa}$$

$$u_2 = z_2 = \pm \beta_3\sqrt{\lambda - 1}.$$

The four points arising by all possible combinations of signs are found to keep the four points on the surface in which it is cut by the line of fixed points $\begin{cases} x_0 - x_2 = 0 \\ x_1 - x_3 = 0 \end{cases}$. The invariants of the mapping plane under these transformations are these: the points ξ_1, ξ_2 and the maps of the four fixed points; the line joining these points, and four invariant points.

If we apply the transformations T_i to the point $\underline{u}$ we obtain four points $\underline{z}$. If we apply the same transformation to the point $\underline{z}$ we obtain identity and three transformations which give three new points $\underline{u}$

$$T_1: \; z_0' = \lambda \beta_3^2 u_0 u_1$$

$$z_1' = (\mu + \lambda) u_0$$

$$z_2' = \lambda \beta_3^2 u_1 u_2$$

$$T_2: \; z_0' = \lambda \beta_2^2 u_0 u_2$$

$$z_1' = \lambda \beta_2^2 u_1 u_2$$

$$z_2' = (\lambda-1)\, u_0$$

$$T_3:\quad z_0' = \lambda\beta_2^2\beta_3^2\, u_1u_2$$

$$z_1' = (\lambda+1)\,\beta_2^2\, u_0u_2$$

$$z_2' = (\lambda-1)\,\beta_3^2\, u_0u_1$$

The singular elements are the points and lines of the reference triangle. The fixed elements are the four points invariant under Π.

These transformations T correspond to the collineations in the surface in the same sense that the transformations Π correspond to the polarities. The eight elements Π_i and T_i with identity form an Abelian G_8 like in all particulars to the involution G_8 of the surface. The multiplication table is here repeated.

	1	T_1	T_2	T_3	Π_1	Π_2	Π_3	Π_4
1	1	T_1	T_2	T_3	Π_1	Π_2	Π_3	Π_4
T_1	T_1	1	T_3	T_2	Π_2	Π_1	Π_4	Π_3
T_2	T_2	T_3	1	T_1	Π_3	Π_4	Π_1	Π_2
T_3	T_3	T_2	T_1	1	Π_4	Π_3	Π_2	Π_1
Π_1	Π_1	Π_2	Π_3	Π_4	1	T_1	T_2	T_3
Π_2	Π_2	Π_1	Π_4	Π_3	T_1	1	T_3	T_2
Π_3	Π_3	Π_4	Π_1	Π_2	T_2	T_3	1	T_1
Π_4	Π_4	Π_3	Π_2	Π_1	T_3	T_2	T_1	1

We have thus shown that the correspondence on the surface, arising from the correlation group, between a point y and a tangent plane with contact x, gives rise in the mapping plane to a group of Cremona transformations which show the same correspondence between the maps of points y and x.

§6. The Anchor Ring.

The equation of the Dupin Cyclide shows two independent constants. The Anchor Ring arises as a special case of the cyclide in which only one constant is independent

Considering the Anchor Ring as generated by the revolution of a circle, radius r, about a fixed line at a given distance a from the center, we refer the figure to rectangular axes in space. Let the y axis be the axis of revolution and take the initial position of the revolving circle in the xz plane, center (0, a). The equation is

$$(59).\quad x^4+y^4+z^4+(r^2-a^2)w^2+2(x^2z^2+x^2y^2+y^2z^2)$$
$$-2(a^2+r^2)(x^2+z^2)w^2-2(r^2-a^2)y^2w^2=0.$$

Here the two constants a^2+r^2 and a^2-r^2 are plainly dependent, if a is taken constant.

This Anchor Ring may be shown

to possess all the characteristic properties of the cyclide. It is a quartic surface with four double points and a double conic in $w=0$. It has also four lines and a cone of double tangent lines.

The existence of the cone of double tangent lines is geometrically evident. It has as elements the tangent lines through the origin which touch the surface along its intersection with $y = \pm\sqrt{a^2-r^2}\cdot\frac{r}{a}$. These two circles correspond to the quartic curve of contacts in the cyclide. The double conic is the absolute conic. Two of the double points are evidently the intersections with the y axis of the revolving circle, real points when $r>a$. The two remaining are the two points in the y plane common to all sections of the surface parallel to the y plane, that is, the circu-

[illegible] points in that plane.

The lines of the cyclide join the double points. The diagonals, not on the surface, joining pairs of double points are these the y axis and the line at infinity in the y plane. The four lines on the surface must be the remaining joins of these points. To find these lines we recall that on the cyclide the lines lay in four planes, common tangents to the cone of double lines and the cone to the double conic. Each passes through two double points and one singular point on the cone of double lines. In the Anchor Ring the cone of double lines is

$$r^2(a^2-r^2)(x^2+y^2+z^2)-(r^2+a^2)a^2y^2 \pm 2a^2r^2y^2=0.$$

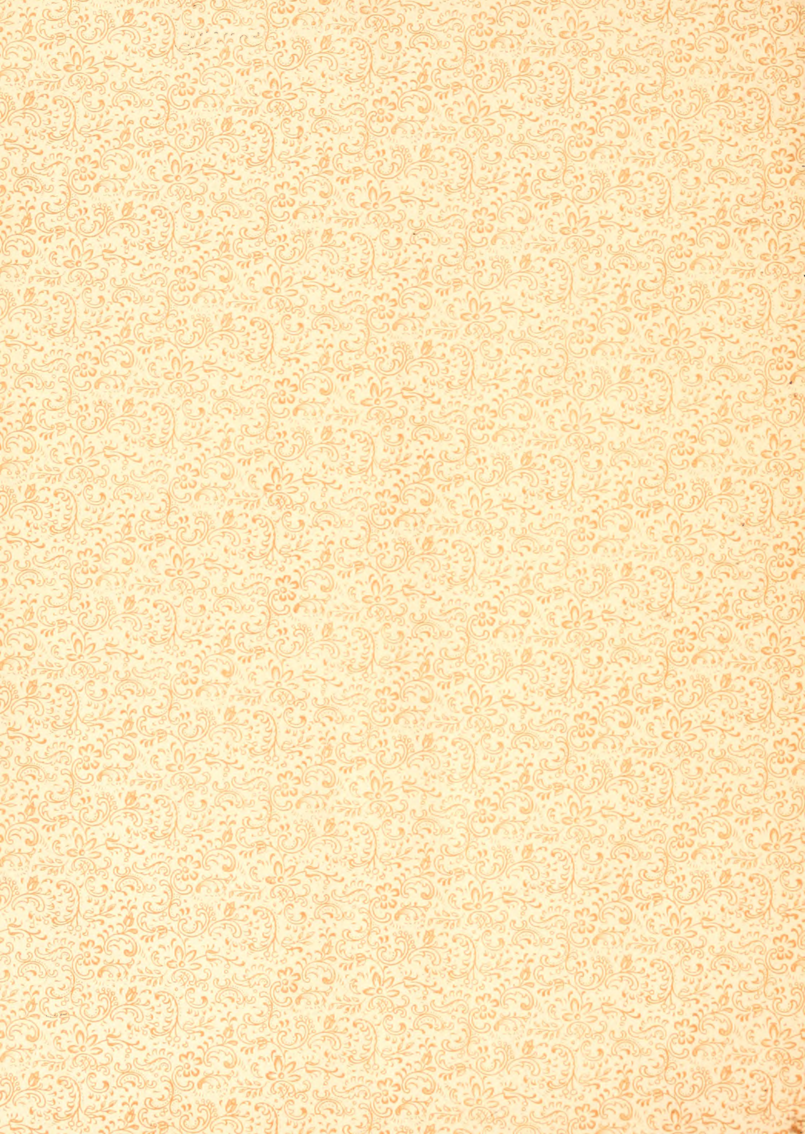
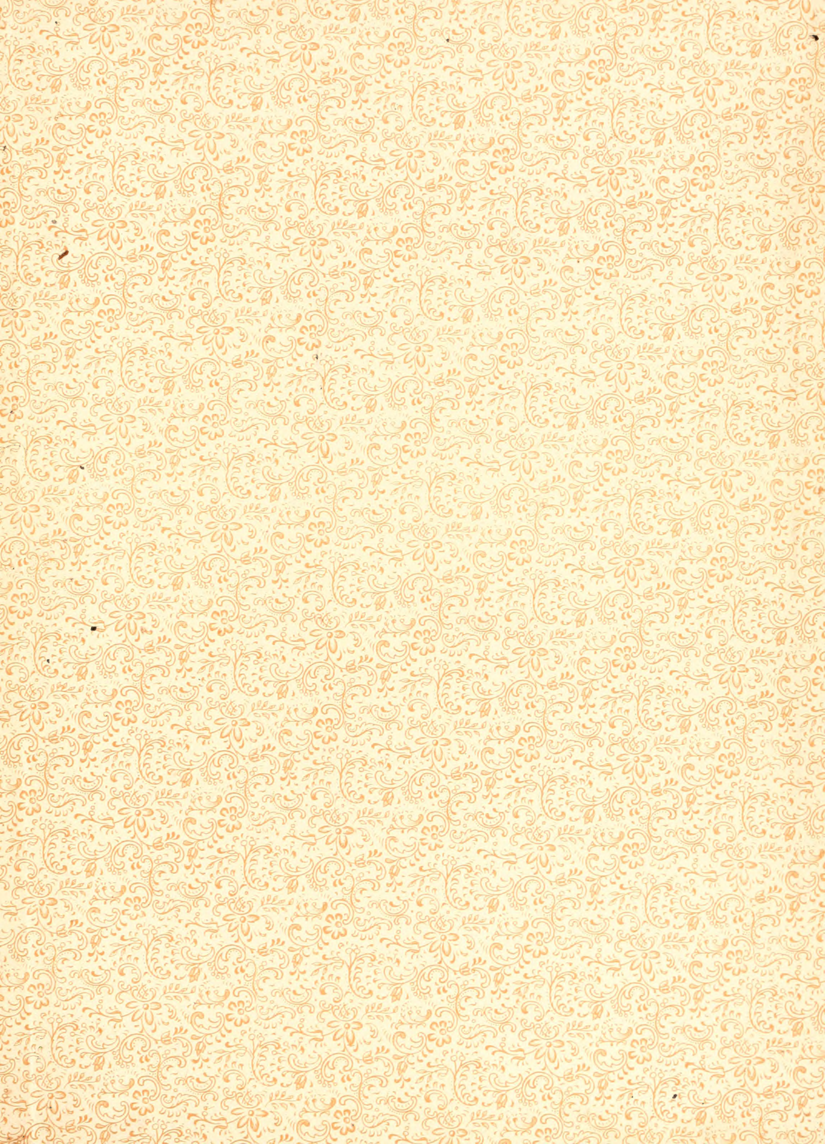
This is of the form $C_1(x^2+y^2+z^2)+C_2y^2=0$ and hence has double contact with

the cone is the double conic, $x^2+y^2+z^2=0$ along the two lines in which the latter is met by $y=0$. The four intersections of the two cones in the plane at infinity, distinct in the case of the cyclide, become here two points of contact, I and J in $y=0$. Common tangent planes of the two cones here touch at these points. I and J are also two singular points on the two circles into which the quartic curve of contact has broken up. The lines of the surface hence lie in planes tangent to the absolute conic at I and J in $y=0$, and join the double points $(0, \pm\sqrt{r^2-a^2}, 0)$ separately to I and J. They lie therefore in $x^2+y^2=0$ and $y^2=r^2-a^2$.

This surface in these results corresponds in all essential respects

to Dupin's Cyclide but is specialised by the breaking of the generating curve, lines of contacts of double tangent lines, which are conics, being in parallel planes and passing through two common points.

Vita.

Mabel Minerva Young was born in Worcester, Massachusetts, in 1873. She attended the public schools of that city. She received the degree B.A. from Wellesley College in 1898. During the year 1898-1899 she was a graduate student in education and psychology at Columbia University, from which she received the degree A.M. and the diploma of Teachers College in 1899. From 1899-1902 she taught at Northfield Seminary and during the years 1904-1909; 1910-1911 was instructor in mathematics at Wellesley College. In 1909 she entered Johns Hopkins University as graduate student in mathematics, physics and education, and has been in residence during the years 1909-1910; 1912-1914.